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**SUMMATION OF LEADING LOGARITHMS AT SMALL  $x$** 

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**Abstract**

We show how perturbation theory may be reorganized to give splitting functions which include order by order convergent sums of all leading logarithms of  $x$ . This gives a leading twist evolution equation for parton distributions which sums all leading logarithms of  $x$  and  $Q^2$ , allowing stable perturbative evolution down to arbitrarily small values of  $x$ . Perturbative evolution then generates the double scaling rise of  $F_2$  observed at HERA, while in the formal limit  $x \rightarrow 0$  at fixed  $Q^2$  the Lipatov  $x^{-\lambda}$  behaviour is eventually reproduced. We are thus able to explain why leading order perturbation theory works so well in the HERA region.

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The evolution of structure functions at small  $x$  is expected to be problematic within the conventional framework [1] of perturbative QCD, in that the usual perturbative evolution appears to be unstable at small  $x$ . This expectation rests on the observation that singlet anomalous dimensions grow in an unbounded way as  $N \rightarrow 0$  [2] (specifically, at order  $m$  in perturbation theory  $\gamma_N^{gg}(\alpha) \sim (\frac{\alpha}{N})^m$ ). Indeed, the leading order term drives a corresponding growth in the singlet structure function as both  $\frac{1}{x}$  and  $Q^2$  increase [3]. However if  $\frac{1}{x}$  increases much more rapidly than  $t \equiv \ln Q^2/\Lambda^2$ , higher order terms become increasingly important, and it is no longer clear whether perturbation theory can be trusted[2]. Hence it is often claimed that the usual evolution equations [1] must then be abandoned in favour of equations [4] which resum the logs of  $\frac{1}{x}$  at fixed  $t$ , resulting in an even stronger (power-like) growth.

Strong scaling violations at large  $Q^2$  accompanied by a corresponding increase at small values of  $x$  are indeed observed in recent measurements of the proton structure at HERA[5,6]. The leading order perturbative prediction [3] can thus now be tested empirically: the structure function data should display a double scaling behavior[7] in the two variables  $\sigma \sim \sqrt{\ln \frac{1}{x} \ln t}$  and  $\rho \sim \sqrt{\ln \frac{1}{x} / \ln t}$ , rising linearly in the former while remaining relatively independent of the latter. This scaling follows directly from the fact that the leading order small- $x$  evolution equations reduce asymptotically to an isotropic wave equation in the plane of  $\ln \frac{1}{x}$  and  $\ln t$ . In fact not only is the scaling observed[7], but even the slope of the rise turns out to be remarkably close to the leading order prediction [8]. These results remain stable upon the inclusion of two loop corrections [9], and it now seems possible to detect two loop effects in the data through a small reduction in the slope.

All this suggests that it should be possible to keep the behaviour of the perturbative expansion under control by summing leading logs of  $\frac{1}{x}$  as well as logs of  $Q^2$ , while remaining within the standard framework of perturbative evolution [10]. The double asymptotic scaling behaviour observed in the HERA data should then emerge as the dominant behaviour in some well determined region of the  $x$ - $t$  plane. Such a programme is possible since the leading singularities in both  $\gamma_N^{gg}$  and  $\gamma_N^{qg}$  have now been explicitly computed to all orders in  $\alpha_s/N$  [11-15]. However in a previous study[16], in which three and four loop singularities were retained, it appeared that instabilities may develop which signal the breakdown of perturbation theory. Here we will explain in detail how the perturbative expansion of anomalous dimensions can be reorganized at small- $x$  in a way which is consistent with the renormalization group, but in which the leading logs of  $\frac{1}{x}$  are summed to all orders. We will then prove explicitly that this leads to stable perturbative evolution equations for

parton distributions at all positive values of  $x$ , in which all corrections are either of higher order in  $\alpha_s$  or are higher twist, and which thus breaks down only at low values of  $Q^2$ . Solution of these equations will then enable us to determine in precisely which region of the  $x$ - $t$  plane double scaling should remain valid, and whether it will be possible to observe the stronger power-like rise characteristic of solutions of the BFKL equation.

According to the operator product expansion and renormalization group, the Mellin moments  $f_N^i(t) \equiv \int_0^1 dx x^N f^i(x; t)$  of the parton distribution functions  $f^i(x; t)$  evolve multiplicatively with  $t$ :

$$\frac{d}{dt} f_N^i(t) = \sum_j \gamma_N^{ij}(\alpha_s(t)) f_N^j(t), \quad (1)$$

where  $\gamma_N^{ij}(\alpha)$  are the anomalous dimensions of local operators [1]. The perturbative evolution then sums all logarithms of  $Q^2$ , so the  $O(\alpha)$  and  $O(\alpha^2)$  terms are known as the leading and subleading log approximations respectively. Perturbative expansion of the anomalous dimensions is justified by factorization theorems; these have been shown recently to apply to all orders even at small  $x$  [13-15]. The inverse Mellin transform of (1) gives

$$\frac{d}{dt} f^i(x; t) = \sum_j \int_x^1 dy P^{ij}(y; t) f^j\left(\frac{x}{y}; t\right), \quad (2)$$

where the splitting functions  $P^{ij}(x; t)$  are related to the anomalous dimensions by<sup>1</sup>

$$\gamma_N^{ij}(\alpha_s(t)) = \int_0^1 dx x^N P^{ij}(x; t). \quad (3)$$

This version of the evolution equation is more physical because it only involves integration over the physically accessible region  $y > x$  [17].

At small  $x$  it is of course necessary to take into account the fact that, if  $\frac{1}{x}$  is large enough, Feynman diagrams which contain powers of  $\ln \frac{1}{x}$  may be just as important as those with powers of  $\ln Q^2$ . It is thus no longer appropriate to organize the perturbative expansion of the splitting functions in powers of  $\alpha_s$  only. In the Mellin space equation (1) these extra logarithms are due to the presence of poles in the anomalous dimensions as  $N \rightarrow 0$ . Now, while nonsinglet anomalous dimensions are regular as  $N \rightarrow 0$ , singlet anomalous dimensions computed at leading order in  $\alpha$  have a  $\frac{1}{N}$  singularity; at  $m$ -th order they have at most (at least in reasonable renormalization schemes) an  $N^{-m}$  singularity

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<sup>1</sup> This definition differs from the usual one by a factor of  $\frac{\alpha_s}{2\pi}$ ; also our moment variable  $N$  differs by one from the usual one so that for us the first moment is that with  $N = 0$ .

[11-15]. Hence, separating out these singularities, and suppressing the flavor indices  $i, j$  for clarity, the anomalous dimensions admit the perturbative expansion (see fig. 1a)

$$\gamma_N(\alpha) = \sum_{m=1}^{\infty} \alpha^m \sum_{n=-\infty}^m A_n^m N^{-n} = \sum_{m=1}^{\infty} \alpha^m \left( \sum_{n=1}^m A_n^m N^{-n} + \bar{\gamma}_N^{(m)} \right), \quad (4)$$

where  $A_n^m$  are simply numerical coefficients, and  $\bar{\gamma}_N^{(m)}$  are all regular as  $N \rightarrow 0$ . This then corresponds to the expansion

$$P(x; t) = \sum_{m=1}^{\infty} (\alpha_s(t))^m \left( \frac{1}{x} \sum_{n=1}^m A_n^m \frac{\ln^{n-1} \frac{1}{x}}{(n-1)!} + \bar{P}^{(m)}(x) \right) \quad (5)$$

of the splitting function, where  $\bar{P}^{(m)}(x)$  are regular as  $x \rightarrow 0$ . At small  $x$  the usefulness of this expansion in powers of  $\alpha$  is spoilt by the logs of  $1/x$  which can compensate for the smallness of  $\alpha_s(t)$ .

However this is not the only way to order the expansion: if instead of expanding in  $\alpha$  and then in  $N$ , as in (4), we choose instead to expand in  $\alpha$  and  $\alpha/N$  (see fig. 1b), the anomalous dimensions take the form

$$\gamma_N(\alpha) = \sum_{m=1}^{\infty} \alpha^{m-1} \left( \sum_{n=2-m}^{\infty} A_n^{n+m-1} \left( \frac{\alpha}{N} \right)^n \right), \quad (6)$$

which corresponds to the splitting function

$$P(x; t) = \sum_{m=1}^{\infty} (\alpha_s(t))^{m-1} \left( \frac{1}{x} \sum_{n=1}^{\infty} A_n^{n+m-1} \frac{(\alpha_s(t))^n \ln^{n-1} \frac{1}{x}}{(n-1)!} + \sum_{q=0}^{m-2} A_{-q}^{m-q-1} (\alpha_s(t))^{-q} \frac{d^q}{dx^q} \delta(1-x) \right). \quad (7)$$

While not so useful at large  $x$ , at small  $x$  this is clearly the most appropriate expansion, since at each order in  $\alpha_s$  all the leading logs of  $1/x$  have been summed up.

Solving the evolution equations with the usual expansion eq. (5) of the splitting functions corresponds at leading order ( $m = 1$ ) to summing up all logs of the form

$$\alpha_s^p (\ln Q^2)^q \left( \ln \frac{1}{x} \right)^r \quad (8)$$

with  $q = p$ , and  $0 \leq r \leq p$ . If instead only the sum of leading singularities eq. (7) (again with  $m = 1$ ) is included in the splitting functions, then solving the evolution equations

sums the leading logs (8) with  $r = p$ , and  $1 \leq q \leq p$ . Of course, if contributions which are higher order in  $\alpha_s$  are included in either expansion of the splitting function, eventually all leading logs of both  $\frac{1}{x}$  and  $Q^2$  are included. However while in the former logs of  $Q^2$  are considered leading, in the latter logs of  $\frac{1}{x}$  are leading, the roles of  $q$  and  $r$  in (8) being interchanged. Both expansions are consistent with the renormalization group in the sense that a change in the scale at which the  $m$ -th order contribution to the expansion is evaluated is equivalent to a change in the  $m + 1$ -th order term.

It is now easy to see how to construct an intermediate “double leading” expansion which includes all leading logs, i.e. one such that perturbative evolution sums all terms with  $1 \leq q \leq p$ ,  $0 \leq r \leq p$ , and  $1 \leq p \leq q+r$ , so that each extra power of  $\alpha_s$  is accompanied by a log of either  $\frac{1}{x}$  or  $Q^2$  or both. To do this, the anomalous dimensions are expanded as (see fig. 1c)

$$\gamma_N(\alpha) = \sum_{m=1}^{\infty} \alpha^{m-1} \left( \sum_{n=1}^{\infty} A_n^{n+m-1} \left( \frac{\alpha}{N} \right)^n + \alpha \bar{\gamma}_N^{(m)} \right). \quad (9)$$

Each subsequent order of the expansion contains an extra power of  $\alpha$  in comparison to the previous one, so the scheme is still consistent with the renormalization group. Of course, any number of renormalization group consistent expansions which interpolate between the double leading one (eq. (9)) and the large  $x$  (eq. (4)) and small  $x$  (eq. (6)) ones can also be constructed. The important point is that if we choose an expansion which is appropriate at small  $x$  (say, the extreme small  $x$  one (6)), then each subsequent order of the expansion is genuinely of order  $\alpha$  as compared to the previous one, all logarithms having been included in the coefficients. When the scale increases, the higher order contributions are then asymptotically small.

The double scaling behaviour described in refs. [7,8], and observed at HERA[5,6], is governed by the pivotal  $m = n = 1$  term in the various expansions (4),(6) and (9), with  $(A_1^1)^{gg} = \gamma^2 \beta_0 / 4\pi$  determining the slope of the rise of  $F_2^p$  at small  $x$  and large  $Q^2$ . The subleading  $A_0^1$  terms determine  $\delta$  (as defined in ref. [7]) and the relative normalization of the quark and gluon distributions, while  $A_2^2 = 0$ : the leading two-loop term  $A_1^2$  is responsible for a small but observable reduction in the steepness of the slope [9]. Thus at the intermediate range of  $x$  and  $t$  currently being explored at HERA the double leading expansion (9) is the most appropriate, and should be used for a detailed comparison to the data. Here we will instead be interested in the small  $x$  limit and will thus henceforth concentrate on the small  $x$  expansions eqns. (6) and (7), which are sufficient to demonstrate the novel features of perturbative evolution in this kinematic regime.

Before proceeding further, we must confront the convergence issue with which we began the paper. Consider first the leading term in the series (6): since both  $\gamma_N^{qq}$  and  $\gamma_N^{gg}$  vanish at leading order, and<sup>2</sup>  $\gamma_N^{gg} = (C_F/C_A)\gamma_N^{gg}$  [14], we consider explicitly

$$\gamma_N^{gg}(\alpha) = \sum_{n=1}^{\infty} (A_n^n)^{gg} \left(\frac{\alpha}{N}\right)^n + O(\alpha) \equiv (4 \ln 2)^{-1} A(\lambda/N) + O(\alpha), \quad (10)$$

where

$$\lambda(\alpha) = 4 \ln 2 \left( \frac{C_A \alpha}{\pi} \right). \quad (11)$$

Now, it is well known (see e.g. ref. [15]) that this series develops a branch-point singularity at  $N = \lambda$  [4], so that while the series converges for  $|N| > \lambda$ , it is not even Borel summable if  $\text{Re } N < \lambda$ . This seems to pose an insurmountable problem for the perturbative approach to small  $x$  evolution: the series which defines the leading coefficient in expansion (6) of the anomalous dimension, which was supposed to be useful for small  $N$ , is actually only well defined when  $N$  is sufficiently large. This apparent inconsistency seems to have led many to the conclusion that conventional perturbative evolution breaks down at small  $x$ .

In fact the dilemma is resolved by the observation that the physically relevant quantity, namely the splitting function  $P^{gg}(x)$ , is instead given at leading order in the expansion (7) by the series

$$P^{gg}(x; t) = \frac{\alpha_s(t)}{x} \sum_{n=1}^{\infty} (A_n^n)^{gg} \frac{(\alpha_s(t) \ln \frac{1}{x})^{n-1}}{(n-1)!} \equiv \frac{C_A}{\pi} \frac{\alpha_s(t)}{x} B(\lambda_s(t) \ln \frac{1}{x}), \quad (12)$$

where now  $\lambda_s(t) \equiv \lambda(\alpha_s(t))$ . This series converges uniformly on any finite intervals of  $x$  and  $t$  which exclude  $x = 0$ . To prove this, we write the two series (10) and (12) as

$$A(v) = \sum_{n=1}^{\infty} a_n v^n, \quad B(u) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} u^{n-1}, \quad (13)$$

where the coefficients  $a_n \equiv 4 \ln 2 (A_n^n)^{gg} (4 \ln 2 C_A / \pi)^{-n}$ . The series for  $A(v)$  then has radius of convergence one, from which it follows trivially that the radius of convergence of  $B(u)$  is infinite.

The defining relation (3) may now be rewritten in the form

$$A(v) = \int_0^{\infty} du e^{-u/v} B(u) : \quad (14)$$

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<sup>2</sup> Following the usual conventions,  $C_A = 3$ ,  $C_F = \frac{4}{3}$ ,  $T_R = \frac{1}{2}$  for QCD with three colors.

$B$  is the Borel transform of  $A$ . The series  $A(v)$  is not Borel summable for  $\text{Re } v^{-1} < 1$  (i.e.  $\text{Re } N < \lambda$ ) because the integral no longer converges at the upper limit (corresponding to the lower limit of the integral over  $x$  in (3)): the real coefficients  $a_n$  are all positive and decrease only very slowly as  $n \rightarrow \infty$ , so  $B(u) \sim e^u$  as  $|u| \rightarrow \infty$ .<sup>3</sup> It follows that the only reason for the bad behaviour of the series (10) is that when transforming to Mellin space one attempts to integrate all the way down to  $x = 0$ , and this is not possible for singlet distributions because the total number of partons diverges there. If instead the parton distributions are evolved using the Altarelli-Parisi equations (2), the splitting functions are only required over the physically accessible region  $x > x_{\min}$  [17], and no convergence problems arise. Indeed for all physical applications it is sufficient to truncate the series (12) after a finite number of terms.<sup>4</sup>

Similar considerations presumably apply to subleading terms in the series (6) and (7). Recently the first nonvanishing term in the large  $N$  expansions of  $\gamma^{qg}$  and  $\gamma^{qq}$  have been calculated [14]: the result may be written in the form

$$\gamma_N^{qg}(\alpha) = T_R \frac{\alpha}{3\pi} (1 + \tilde{A}(\lambda/N)) + O(\alpha), \quad \gamma_N^{qq}(\alpha) = \frac{C_F T_R}{C_A} \frac{\alpha}{3\pi} \tilde{A}(\lambda/N) + O(\alpha), \quad (15)$$

where  $\tilde{A}(v)$  has the same form as  $A(v)$  (13), with new coefficients  $\tilde{a}_n$ , but the same radius of convergence. It follows immediately that the corresponding expressions for  $P^{qg}(x; t)$  and  $P^{qq}(x; t)$  are convergent for all nonzero  $x$ . It is tempting to conjecture that the same must be true for all the coefficients in the small  $x$  expansion (7): all that is necessary is that the coefficients in the expansion of the corresponding anomalous dimensions (6) have a positive radius of convergence, i.e. that there is no singularity (or accumulation of singularities) at  $N = \infty$ , or equivalently at  $\alpha = 0$ . This is a conventional assumption in perturbative QCD.

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<sup>3</sup> For timelike anomalous dimensions, relevant for the small  $x$  evolution of fragmentation functions, the corresponding series have alternating coefficients, and thus converge throughout the physical region. The same is true of  $\phi_6^3$  theory.

<sup>4</sup> In practice this means that when solving the evolution equations (1) numerically, the divergence of the series expansion of the anomalous dimensions at small  $N$  can simply be ignored: although adding another term in the series (10) can change the anomalous dimension a lot at small  $N$ , this will be compensated by a corresponding shift in the steepest descent contour, such that the change in the evolved distribution is actually very small. We have checked this numerically.

Using the analytic expressions given in [11,14] we have computed the coefficients  $a_n$  and  $\tilde{a}_n$  to very high orders. In practice only the first dozen or so are needed in any realistic calculation: we give the first thirty six in the table.

We are now finally able to write down evolution equations for parton distribution in the leading order of the small  $x$  expansion of the splitting functions (7). Because only the  $P^{gg}$  and  $P^{gq}$  splitting functions are nonvanishing at leading order, only the Altarelli-Parisi equation satisfied by the gluon distribution survives; the determination of the quark distribution, and thus of structure functions, requires the inclusion of subleading corrections and will be discussed at the end of the paper. As in [7] the evolution equation may be simplified by introducing the variables

$$\xi \equiv \ln \left( \frac{x_0}{x} \right), \quad \zeta \equiv \ln \left( \frac{t}{t_0} \right) = \ln \left( \frac{\ln Q^2 / \Lambda^2}{\ln Q_0^2 / \Lambda^2} \right), \quad (16)$$

and defining  $G(\xi, \zeta) \equiv xg(x; t)$ . Using the splitting function (12) in the evolution equation (2) then gives

$$\begin{aligned} \frac{\partial}{\partial \zeta} G(\xi, \zeta) &= \frac{4C_A}{\beta_0} \sum_{n=1}^{\infty} a_n \frac{\lambda_s(\zeta)^{n-1}}{(n-1)!} \int_{-\xi_0}^{\xi} d\xi' (\xi - \xi')^{n-1} G(\xi', \zeta) \\ &= \gamma^2 \sum_{n=0}^{\infty} a_{n+1} \lambda_s(\zeta)^n \int_{-\xi_0}^{\xi} d\xi_1 \int_{-\xi_0}^{\xi_1} d\xi_2 \dots \int_{-\xi_0}^{\xi_n} d\xi' G(\xi', \zeta). \end{aligned} \quad (17)$$

Here  $\gamma \equiv \sqrt{4C_A/\beta_0}$  (as in [7]): we have used the one loop form of  $\alpha_s(t) = \frac{4\pi}{\beta_0 t}$ ,  $\beta_0 = 11 - \frac{2}{3}n_f$ , as is appropriate for a leading order calculation. Since we retain only singular contributions to the splitting functions the lower limit  $\xi_0 = \ln \frac{1}{x_0}$  in the integrations on the right hand side can be consistently set to zero.

Note that if the large  $\xi$  (i.e. small  $x$ ) limit is approached by letting  $\xi$  grow as  $\zeta^k$  with any  $k > 1$ , but more slowly than  $e^\zeta$ , corrections to the leading order splitting function are exponentially suppressed. If, however, we let  $\xi$  grow faster than  $e^\zeta$  (i.e.  $1/x$  grow faster than a power of  $Q^2$ ), then higher orders in the leading singularity expansion of the splitting function will eventually become significant, growing as  $e^{\xi \lambda_s(\zeta)}$ .

Differentiating both sides with respect to  $\xi$ , the gluon evolution equation eq. (17) can be cast in the form of a wave equation as in ref.[7]:

$$\begin{aligned} \frac{\partial^2}{\partial \xi \partial \zeta} G(\xi, \zeta) &= \gamma^2 G(\xi, \zeta) + \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \lambda_s(\zeta)^n \int_0^{\xi} d\xi_1 \int_0^{\xi_1} d\xi_2 \dots \int_0^{\xi_{n-1}} d\xi' G(\xi', \zeta) \\ &= \gamma^2 G(\xi, \zeta) + \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \frac{\lambda_s(\zeta)^n}{(n-1)!} \int_0^{\xi} d\xi' (\xi - \xi')^{n-1} G(\xi', \zeta). \end{aligned} \quad (18)$$



If only the first leading singularity is retained, then eq. (18) is simply the wave equation discussed in ref. [7] (with  $\delta = 0$ ; subleading corrections have not yet been included), and leads directly to a double scaling behaviour of the gluon distribution when the boundary conditions are sufficiently soft. When the higher singularities are also included on the right hand side, then the generic features of the propagation in the  $\xi$ - $\zeta$  plane due to the wave-like nature of the evolution are preserved. In particular, the propagation is still linear and causal, and far from the boundaries independent of the detailed form the boundary condition.

For any reasonable values of  $\xi$  it is clear that only the first few terms in the expansion of eq. (12) will contribute (notice that in fact  $a_2 = a_3 = a_5 = 0$ ), and then the asymptotic form of the solution will be essentially unchanged (i.e., it will display double asymptotic scaling). The easiest way of seeing this explicitly is to recall that the leading order solution in which only the first term on the right hand side is retained is just given by a Bessel function  $G_0(\xi, \zeta) \sim I_0(2\gamma\sigma)$ , where, as in [7], we find it useful to define the scaling variables  $\sigma \equiv \sqrt{\xi\zeta}$ ,  $\rho \equiv \sqrt{\xi/\zeta}$ . The asymptotic behaviour  $I_0(z) \sim z^{-1/2}e^z$  as  $z \rightarrow \infty$  then gives rise to double asymptotic scaling. If this is substituted back into the new terms on the right hand side, to generate the first term in an iterative solution, we may use the fact that  $z^n I_{n-1}(z) = \frac{d}{dz}(z^n I_n(z))$  to write (18) in the approximate form

$$\frac{\partial^2}{\partial \xi \partial \zeta} G(\xi, \zeta) \simeq \gamma^2 G(\xi, \zeta) + \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \left( \frac{\rho \lambda_s(\zeta)}{\gamma} \right)^n I_n(2\gamma\sigma). \quad (19)$$

Since  $\rho^n I_n(2\gamma\sigma)$  is bounded above by  $\xi^n I_0(2\gamma\sigma)$  it follows that double scaling will always set in asymptotically provided the series  $\sum_{n=1}^{\infty} a_{n+1} \left( \frac{\xi \lambda_s(\zeta)}{\gamma} \right)^n$  converges uniformly, that is provided  $\xi < \frac{\gamma}{\lambda_s(\zeta)}$ .

However for very large  $\xi$  the splitting function will eventually be dominated by the higher orders of the series eq. (12). This is the limit which is often approached by means of the BFKL equation [4], which resums all leading logs of  $\frac{1}{x}$ , i.e. all terms with  $p = r$ ,  $q = 0$  in (8), but with the  $Q^2$  dependence of  $\alpha_s$  suppressed. Since at leading order eq.(18) already includes all the leading twist information contained in the BFKL equation (but with all higher twist and infrared behaviour factored out), it should reproduce the same power-like rise in the limit  $\xi \rightarrow \infty$ , provided that we freeze the coupling. We can both check this, and understand how the behaviour of the gluon evolution changes when the coupling runs, by taking advantage of the fact that when the series eq. (12) is dominated by its higher order terms, the evolution equation eq. (18) takes a simple closed form.

Because the series (10),(13) has unit radius of convergence it follows that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \rightarrow 1$ .<sup>5</sup> But then setting  $a_{n+1} \approx a_n$  in the sum in eq. (18), shifting the summation index by one unit, and using eq. (17), we have

$$\gamma^2 \sum_{n=1}^{\infty} a_{n+1} \lambda_s(\zeta)^n \int_0^\xi d\xi_1 \int_0^{\xi_1} d\xi_2 \dots \int_0^{\xi_{n-1}} d\xi' G(\xi', \zeta) = \lambda_s(\zeta) \frac{\partial G}{\partial \zeta}. \quad (20)$$

Hence, asymptotically as  $\xi \rightarrow \infty$  the evolution equation (18) becomes simply

$$\frac{\partial^2}{\partial \xi \partial \zeta} G(\xi, \zeta) - \lambda_s(\zeta) \frac{\partial G}{\partial \zeta} = \gamma^2 G(\xi, \zeta), \quad (21)$$

and asymptotically the summation of all leading singularities leads to a damping term in the wave equation.<sup>6</sup>

It is now trivial to recover the singular ‘Lipatov pomeron’ behaviour: if the coupling is frozen, then  $\lambda_s$  is just a constant,  $\lambda$ , and the solution to eq. (21) is given in terms of the (double scaling) solution  $G_0(\xi, \zeta)$  to the original wave equation [7]

$$G(\xi, \zeta) = e^{\lambda \xi} G_0(\xi, \zeta) = x^{-\lambda} G_0(\xi, \zeta). \quad (22)$$

The solution with fixed coupling thus displays a strong power-like growth [4] in the region  $\xi \gg \frac{\gamma^2}{\lambda^2} \zeta$ , while below this line we have the usual scaling behaviour (fig. 2a). This situation is very similar to that obtained by solving the original  $\lambda = 0$  equation with a hard boundary condition  $G(\xi, 0) \sim e^{\lambda \xi}$  [18,7,16]; however now the hard behaviour is generated perturbatively, by the singularity at  $N = \lambda$  in the anomalous dimension eq. (10), rather than imposed by hand.

Even though (21) may be solved exactly for fixed coupling, it will be useful for the sequel to derive the asymptotic behaviour of the solution (22), using the usual technique of Laplace transformation and steepest descent. Defining  $\mathcal{G}(s, \zeta) \equiv \int_0^\infty d\xi e^{-s\xi} G(\xi, \zeta)$ , (21) may be written as

$$\frac{\partial}{\partial \zeta} \mathcal{G}(s, \zeta) = \frac{\gamma^2}{s - \lambda} \mathcal{G}(s, \zeta) \quad (23)$$

which with  $\lambda$  fixed integrates to

$$\ln \mathcal{G}(s, \zeta) = \ln \mathcal{G}(s, 0) + \frac{\gamma^2}{s - \lambda} \zeta. \quad (24)$$

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<sup>5</sup> It can be seen from the table that this asymptotic form begins to set in only for  $n \sim 20$  however.

<sup>6</sup> As was anticipated, but not proven, in ref. [7].

Estimating the Bromwich integral by steepest descent, the saddle point is located at  $s_0 = \lambda + \gamma/\rho$ , and thus for a soft boundary condition  $G(\xi, 0) \sim 1$  we find that as  $\xi \rightarrow \infty$

$$G(\xi, \zeta) \sim \frac{\gamma/\rho}{\lambda + \gamma/\rho} \frac{1}{\sqrt{\sigma}} e^{2\gamma\sigma + \lambda\xi}. \quad (25)$$

For a hard boundary condition  $G(\xi, 0) \sim x^{-\lambda}$  the first factor would be absent: when leading singularities are included, the form of the boundary condition is only important very close to the boundary (unless of course the boundary condition were even harder than  $x^{-\lambda}$ ). This feature persists in solutions to the full equation (18).

Of course there is really no justification at all for freezing the coupling: we must solve eq. (21) with running coupling  $\lambda(\zeta) = \lambda_0 e^{-\zeta}$ . Since an exact solution is no longer available, we work with the with the Laplace transformed equation (23), which now integrates to

$$\ln \mathcal{G}(s, \zeta) = \ln \mathcal{G}(s, 0) + \frac{\gamma^2}{s} \left( \zeta + \ln \left( \frac{s - \lambda(\zeta)}{s - \lambda_0} \right) \right), \quad (26)$$

and we find a branch cut from  $\lambda(\zeta)$  to  $\lambda_0$ . The saddle point condition leads to a transcendental equation, so we now treat the large and small  $\xi$  limits separately. When  $s$  is large, we can ignore the cut, expand out the logarithm, and find a saddle point at  $s_0 = \frac{\gamma}{\rho} - \frac{\lambda_0 - \lambda}{\zeta} + \dots$ , which gives the asymptotic behaviour

$$G(\xi, \zeta) \sim \frac{1}{\sqrt{\sigma}} e^{2\gamma\sigma + (\lambda_0 - \lambda(\zeta))\rho^2}, \quad (27)$$

i.e. double scaling up to a small correction. This holds throughout the region where the correction is small, i.e. for  $\xi \ll \frac{\gamma^2}{(\lambda_0 - \lambda)^2} \zeta^3$ . The behaviour for large  $\xi$  is found by dominating the Bromwich integral by the branch point singularity at  $s = \lambda_0$ : the saddle point is then at  $s_0 = \lambda_0 + \frac{\gamma^2}{\lambda_0 \xi} + \dots$ , which gives

$$G(\xi, \zeta) \sim \frac{1}{\xi} (\xi(\lambda_0 - \lambda(\zeta)))^{\gamma^2/\lambda_0} e^{\xi\lambda_0 + \gamma^2\zeta/\lambda_0}. \quad (28)$$

Again this behaviour is valid whenever the corrections to it are small, which means that  $\xi \gg \frac{\gamma^2}{\lambda_0^2} e^\zeta$ . Thus the running of the coupling leaves the exponent of the power-like growth unchanged (at  $\lambda_0$ ), but severely limits the region in which this is the dominant behaviour (see fig. 2b). The region in which the scaling behaviour (27) obtains is correspondingly increased.

In practice the damped wave equation (21) is still a rather poor approximation to the full equation (18), since, as may be seen from the table, the ratio  $\frac{a_n}{a_1} \ll 1$  for all  $n > 1$ .

A better approximation may thus be obtained by setting  $a_n = \epsilon a_1$ ; this gives the rather more complicated equation

$$\left(\frac{\partial}{\partial \xi} - \lambda\right) \left(\frac{\partial^2}{\partial \xi \partial \zeta} - \gamma^2\right) G(\xi, \zeta) = \epsilon \gamma^2 \lambda G(\xi, \zeta), \quad (29)$$

which however on Laplace transformation becomes simply

$$\frac{\partial}{\partial \zeta} \mathcal{G}(s, \zeta) = \frac{\gamma^2}{s} \left(1 + \frac{\epsilon \lambda}{s - \lambda}\right) \mathcal{G}(s, \zeta). \quad (30)$$

The asymptotic behaviour of the solution may be found just as above: for running coupling (30) again integrates to (26), but with the logarithm now suppressed by a factor of epsilon, which means that the double scaling solution (27) holds throughout the even larger region  $\xi \ll \frac{\gamma^2}{\epsilon^2(\lambda_0 - \lambda)} \zeta^3$ , the subleading term in the exponent being suppressed by  $\epsilon$ . The power-like growth (28) is now confined to the region  $\xi \gg \frac{\epsilon \gamma^2}{\lambda_0(\lambda_0 - \lambda)} e^{\zeta/\epsilon}$  (see fig. 2c). Since  $\epsilon \lesssim 0.1$ , this means that unless  $\zeta$  is very small, the Lipatov growth only sets in when  $\xi$  is exceedingly large.<sup>7</sup>

These results are confirmed by a numerical analysis of the full evolution equation Eq. (18). The equation was solved by using the integrated form

$$\begin{aligned} G(\xi, \zeta) = G_0(\xi, \zeta) &+ \int_0^\xi d\xi' \int_0^\zeta d\zeta' I_0(2\gamma \sqrt{(\xi - \xi')(\zeta - \zeta')}) \\ &\times \gamma^2 \sum_{n=1}^{\infty} a_{n+1} \frac{\lambda_s(\zeta')^n}{(n-1)!} \int_0^{\xi'} d\xi'' (\xi' - \xi'')^{n-1} G(\xi'', \zeta'), \end{aligned} \quad (31)$$

where  $G_0(\xi, \zeta)$  is the solution of the leading order equation, as given in [7]. In practice the series on the right hand side converges to an accuracy of less than one per cent after only a dozen or so terms. Eqn.(31) may be solved by the standard iterative procedure: convergence is achieved in practice after only a few iterations. For definiteness we chose  $n_f = 4$ ,  $\Lambda = 120\text{MeV}$  (so that  $\alpha_s(M_z) = 0.116$ ), soft boundary conditions  $G(\xi, 0) = G(0, \zeta) = \text{const.}$  and  $x_0 = 0.1$ ,  $Q_0 = 1\text{GeV}$ . The result is shown in the contour plot fig. 3a, to be compared with the leading order plot fig. 3a of ref.[7]. Even though the range of  $\xi$  has been increased, there are no significant scaling violations except very close to the left hand axis.

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<sup>7</sup> For example if  $\zeta \sim 0.4$  (corresponding to  $Q^2$  around  $10\text{GeV}^2$ ), we would need  $\xi$  to be of order 100.

In summary, our analysis of the small  $x$  all-order gluon evolution equation eq. (18) shows that the leading order double scaling behaviour [7] holds asymptotically throughout most of the  $\xi$ - $\zeta$  plane, the power-like singular behavior eq. (22) being confined to a very small wedge close to the boundary  $\zeta = 0$ . This is to be contrasted with the situation for fixed coupling [4], or the phenomenological approach in which a singular behaviour is input to the leading order evolution equation [18,7,16]: in both cases the power like singularity is preserved by the evolution and propagates into a large region  $\xi \lesssim \frac{\gamma^2}{\lambda^2} \zeta$ . The reason that the region in which the power-like growth develops is in reality so small and narrow is basically twofold: the running of the coupling means that the growth near the boundary is rapidly damped away by the exponential fall of  $\alpha_s$  as  $\zeta$  increases, and the size of the growth is in any case severely limited by the smallness of the coefficients  $a_n$  of the leading singularities. It thus seems extremely unlikely that the ‘hard pomeron’ will be seen in structure functions, since at any reasonable value of  $Q^2$ , the power-like rise only sets in at extremely small values of  $x$ ,  $\ln \frac{1}{x} \gg \left( \frac{\alpha_s(t_0)}{\alpha_s(t)} \right)^{1/\epsilon}$ ,  $1/\epsilon \gtrsim 20$ , and in any case such small values of  $x$  are well inside the recombination region [19,15]  $\ln \frac{1}{x} \gg \ln \frac{1}{x_r} \left( \frac{1}{\alpha_s(t)} \right)^2$ ,  $x_r \lesssim 10^{-4}$  where perturbation theory has already broken down, and the rise is presumably damped by the nonperturbative effects necessary to restore unitarity [3].

Having found the gluon distribution, we may now discuss the determination of the quark distribution and the structure function  $F_2(x; t)$ , given in the parton scheme by

$$F_2^p(x; t) = \frac{5}{18} Q(x; t) + F_2^{NS}(x; t)$$

$$Q(x; t) \equiv x \sum_{i=1}^{n_f} (q_i(x; t) + \bar{q}_i(x; t)). \quad (32)$$

In the small  $x$  region, the nonsinglet contribution  $F_2^{NS}(x; t)$  may be neglected. Since  $\gamma^{gg}$  and  $\gamma^{qq}$  begin at order  $\alpha$  (see (15)) to calculate  $Q$  from  $G$  we must go to next-to-leading order in  $\alpha$  in the expansion eq. (6). Writing this as  $\gamma^{ij}(\alpha) = \sum_{a=0}^{\infty} \alpha^a \gamma_a^{ij}$ , we can diagonalize  $\gamma^{ij}(\alpha)$  order by order in  $\alpha$ . To first order the eigenvalues are

$$\lambda^+(\alpha) = \gamma_0^{gg} + \alpha \left( \gamma_1^{gg} + \frac{\gamma_0^{gq}}{\gamma_0^{gg}} \gamma_1^{gq} \right) + O(\alpha^2)$$

$$\lambda^-(\alpha) = \alpha \left( \gamma_1^{qq} - \frac{\gamma_0^{gq}}{\gamma_0^{gg}} \gamma_1^{gq} \right) + O(\alpha^2). \quad (33)$$

The corresponding eigenvectors are given by

$$Q^+ = \alpha \frac{\gamma_1^{gq}}{\gamma_0^{gg}} G^+ + O(\alpha^2) \quad Q^- = -\frac{\gamma_0^{gq}}{\gamma_0^{gg}} G^- + O(\alpha). \quad (34)$$

The eigenvectors are not orthogonal, because  $\gamma^{ij}$  is not symmetric. At leading order the larger eigenvalue  $\lambda^+$  leads to the wave equation (18) for  $G^+$ , while the small eigenvalue vanishes, so  $G^-$  and  $Q^-$  do not evolve. At order  $\alpha$ , (15) gives  $\lambda^-(\alpha) = -\frac{\alpha}{3\pi} \frac{C_F T_R}{C_A}$ , so  $G_N^-(t) = G_N^-(0)e^{-\delta'\zeta}$ ,  $Q_N^-(t) = Q_N^-(0)e^{-\delta'\zeta}$ , where  $\delta' = \frac{16n_f}{27}\beta_0^{-1}$ . The full correction to  $\lambda^+$  is as yet unknown, since only the first two terms of  $\gamma_1^{gg}$  have been computed. Retaining only the first (one loop) term the evolution equation (18) is supplemented by a damping term proportional to  $\delta \equiv (11 + \frac{2n_f}{27})/\beta_0$  as in [7]. The solutions  $G^+(\zeta)$  and  $Q^+(\zeta)$  then acquire an extra factor of  $e^{-\delta\zeta}$ .

Using the eigenvalues (33) and eigenvectors (34), and the expansion (15) for  $\gamma_1^{gg}$ , it is now not difficult to derive the quark evolution equation: taking as boundary conditions  $G(\xi, 0) \equiv G_0(\xi)$ ,  $Q(\xi, 0) \equiv Q_0(\xi)$ ,

$$\begin{aligned} \frac{\partial}{\partial \zeta} Q(\xi, \zeta) = & -\delta' Q_0(\xi) e^{-\delta'\zeta} + \frac{n_f}{9} \gamma^2 e^{-\delta\zeta} \left[ \bar{G}(\xi, \zeta) \right. \\ & \left. + \sum_{n=1}^{\infty} \tilde{a}_n \lambda_s(\zeta)^n \int_0^\xi d\xi' \frac{(\xi - \xi')^{n-1}}{(n-1)!} \bar{G}(\xi', \zeta) \right], \end{aligned} \quad (35)$$

where the gluon distribution  $\bar{G}(\xi, \zeta)$  is a solution of the wave equation (18) with boundary condition  $\bar{G}(\xi, 0) = G_0(\xi) + \frac{4}{9} Q_0(\xi)$ . This equation may then be readily integrated to give the quark distribution and thus  $F_2^p$ . Although we have derived this equation in the parton scheme, it is actually scheme independent: changes in the renormalization scale only affect terms which are of higher order in  $\alpha_s$ .<sup>8</sup> In this sense the expression (35) is still leading order, and should thus be treated on the same footing as (18); scheme dependent subleading corrections first come in through higher loop singularities in  $\gamma_1^{gg}$ .

We may now discuss the asymptotic behaviour of the quark distribution. Since throughout most of the  $\xi$ - $\zeta$  plane  $\bar{G}(\xi, \zeta) \simeq \mathcal{N} I_0(2\gamma\sigma)$ , we may use the same trick as we did to derive (19) to write (35) as (setting  $\delta$  and  $\delta'$  to zero for simplicity)

$$\frac{\partial}{\partial \zeta} Q(\xi, \zeta) \simeq \frac{n_f}{9} \gamma^2 \mathcal{N} \sum_{n=0}^{\infty} \tilde{a}_n \left( \frac{\rho \lambda_s(\zeta)}{\gamma} \right)^n I_n(2\gamma\sigma). \quad (36)$$

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<sup>8</sup> This may be proven explicitly; although in a different scheme (such as  $\overline{\text{MS}}$ ) the coefficients  $\tilde{a}_n$  will be modified, the difference is made up by the new  $O(\alpha_s)$  corrections to the coefficient functions [14].

It follows that when  $\xi \ll \frac{\gamma^2}{\lambda_0^2} \zeta e^{2\zeta}$ ,  $Q(\xi, \zeta)$  scales just as observed experimentally [7,8]. For  $\xi \gg \frac{\gamma^2}{\lambda_0^2} \zeta e^{2\zeta}$ , if we set  $\tilde{a}_n = 1$  for all  $n$ , (36) may be simplified yet further by using the relation  $\sum_{-\infty}^{\infty} t^n I_n(z) = \exp\left(\frac{1}{2}z(t + t^{-1})\right)$ , to give

$$Q(\xi, \zeta) \sim \mathcal{N} \zeta e^{\xi \lambda_0 + \gamma^2 \zeta / \lambda_0}. \quad (37)$$

This result may be readily confirmed using Laplace transforms with respect to  $\xi$ .

It follows that even when the gluon scales,  $F_2$  can still grow rapidly sufficiently close to the  $\zeta = 0$  boundary because of higher order singularities in the anomalous dimension  $\gamma_N^{gg}$ . Indeed these singularities turn out to be much more important than those of  $\gamma_N^{gg}$ , despite being suppressed by an additional power of  $\alpha$ , essentially because the coefficients  $\tilde{a}_n \gg a_n$ . Note that in the BFKL equation [4] quark loops are suppressed, so the contribution of these singularities is not included. In fact even if it were possible to detect the tail of a power-like rise in the gluon distribution, in the structure function this would be masked by a similar rise generated by the subleading singularities in the coupling of the gluons to quarks. This effect may be confirmed numerically by evaluating (35), using as input the gluon distribution displayed in fig. 3a, and taking as boundary condition  $Q_0 \simeq \frac{1}{2}G_0$ . The resulting contour plot of  $\ln F_2$  is displayed in fig. 3b. Indeed  $F_2$  differs very little from the double scaling result except in the wedge close to the left hand boundary, where instead the behaviour (37) is found.

The growth (37) should not be taken too literally, however, since for any reasonable value of  $x$  it only occurs for values of  $Q^2$  very close to the starting scale  $Q_0^2$ , at which nonperturbative effects are also relatively important. Furthermore, in this region higher twist recombination effects are expected to become significant [19]. More interestingly, the higher order terms in (35) increase the relative normalization of  $Q$  with respect to  $G$  at small  $x$  for a given starting scale  $Q_0$  (by around a factor of two). Since when  $Q_0 = 1$  GeV the leading order result has approximately the correct normalization, this suggests that at all orders the starting scale  $Q_0$  at which the soft pomeron boundary condition is imposed should be raised to compensate. In fact, a starting scale of around 2 GeV turns out to be required; the resulting contour plot of  $\ln F_2$  is shown in fig. 3c. Since  $\lambda_0$  is a little lower, the scaling behaviour is also improved, the power-like growth being now confined to the top left-hand corner of the plot.

To make the comparison with the double scaling results of ref. [8] more quantitative, we may compute the slope of the linear rise of (in the notation of [7,8])  $\ln R'_F F_2$  in the

HERA region. In leading order this is simply given by  $2\gamma = 2.4$ , which is reduced by around 20% by two loop corrections [9], whereas the HERA data give  $2.4 \pm 0.2$ . The corresponding slopes for the three distributions plotted in fig. 3 are 2.5, 2.9 and 2.6 respectively. Clearly a detailed fit to the data would thus require the inclusion of all of these competing effects.

In conclusion, the leading twist evolution equation which sums all leading logarithms at small  $x$ , but is also fully consistent with the renormalization group, interpolating smoothly between small  $x$  and large  $x$ , is just the Altarelli-Parisi equation (2), with the splitting function expanded in powers of  $\alpha_s(t)$ , but retaining terms to all orders in  $\alpha_s(t) \ln \frac{1}{x}$ . Perturbation theory at small  $x$  is then placed on the same footing as perturbation theory at large  $x$ : there are no instabilities or problems of convergence. In fact deep inelastic scattering at small  $x$  and large  $Q^2$  is a much cleaner perturbative environment than more traditional deep inelastic scattering at large  $x$ , since the non-perturbative component of the structure function is relatively insignificant [7,9]. Taken altogether, the analysis presented here explains why the simple double asymptotic scaling picture presented in [7], though only at leading (and subleading [9]) order in  $\alpha_s$ , accounts so well [8] for the rise in  $F_2^p$  seen at HERA [5,6]. Perturbation theory breaks down only when higher twist effects become necessary to restore unitarity.

Although they have little effect on the qualitative form of  $F_2$  for any reasonable values of  $Q^2$ , the higher order logs in  $P^{qg}$  do make a significant impact on the relative normalization of the quark and gluon distributions. This means that attempts to extract the gluon distribution from  $F_2$  using the standard two-loop evolution equation should be treated with caution: an all-order calculation could give a gluon distribution which is smaller by as much as a factor of two. The summation of subleading logarithms will become increasingly important as the quality and range of structure function data at small  $x$  improves further.

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## Figure Captions

- Fig. 1. The terms summed in the various expansions of the anomalous dimensions and associated splitting functions: a) the standard (large- $x$ ) expansion (4), b) the small- $x$  expansion (6), and c) the ‘double-leading’ expansion (9). Leading, sub-leading and sub-sub-leading terms are indicated by the solid, dashed and dotted lines respectively;  $m$  denotes the order in  $\alpha$ , while  $n$  denotes the order in  $1/N$ . Singular terms are marked as crosses, while terms whose coefficients known at present (for  $\gamma_N^{gg}$ ) are marked by circles: the term which leads to double scaling is marked with a star.
- Fig. 2. Regions of validity in the  $\xi$ - $\zeta$  plane of various asymptotic behaviours: a) from the damped wave equation (21) with fixed coupling, b) the same with running coupling, and c) from the equation (29) which takes into account the smallness of the coefficients  $a_n$ . The region marked ‘S’ denotes the double scaling region, while ‘L’ denotes the Lipatov (power rise) region, and ‘R’ the recombination region.
- Fig. 3. Contour plots of the logarithms of the solutions of the small- $x$  evolution equations in the  $\xi$ - $\zeta$  plane: a) the gluon distribution  $G$ , b) the structure function  $F_2$  with  $Q_0 = 1$  GeV, and c) the same but with  $Q_0 = 2$  GeV. The values of the other parameters are explained in the text. The contours are all equally spaced: the interval between adjacent contours is  $\Delta \ln G = 0.85$ ,  $\Delta \ln F_2 = 0.62, 0.58$  respectively. A scatter plot of the HERA data[5,6] is superimposed for reference.

$n$	$a_n$	$\tilde{a}_n$
1	1	0.78145981
2	0	0.29913310
3	0	0.38806675
4	0.11279729	0.25256339
5	0	0.17838311
6	0.01265760	0.22632345
7	0.03816969	0.15473331
8	0.00160119	0.13891462
9	0.01142194	0.15744938
10	0.01742873	0.11602071
11	0.00260717	0.11532868
12	0.00888439	0.11997040
13	0.00942680	0.09568020
14	0.00300214	0.09846280
15	0.00670932	0.09707906
16	0.00581292	0.08291446
17	0.00301620	0.08559381
18	0.00507784	0.08203738
19	0.00400435	0.07384063
20	0.00283184	0.07550427
21	0.00390386	0.07157467
22	0.00301599	0.06682914
23	0.00256389	0.06747701
24	0.00306948	0.06392641
25	0.00242542	0.06112241
26	0.00227736	0.06101759
27	0.00247526	0.05808061
28	0.00203856	0.05633377
29	0.00200551	0.05576012
30	0.00204715	0.05343445
31	0.00176238	0.05224401
32	0.00176276	0.05142727
33	0.00173286	0.04961887
34	0.00155085	0.04871349
35	0.00155308	0.04780818
36	0.00149656	0.04640235

Table: The coefficients  $a_n$  and  $\tilde{a}_n$ , computed using formulae in ref.[11] and ref.[14] respectively.



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<http://arxiv.org/ps/hep-ph/9501231v3>

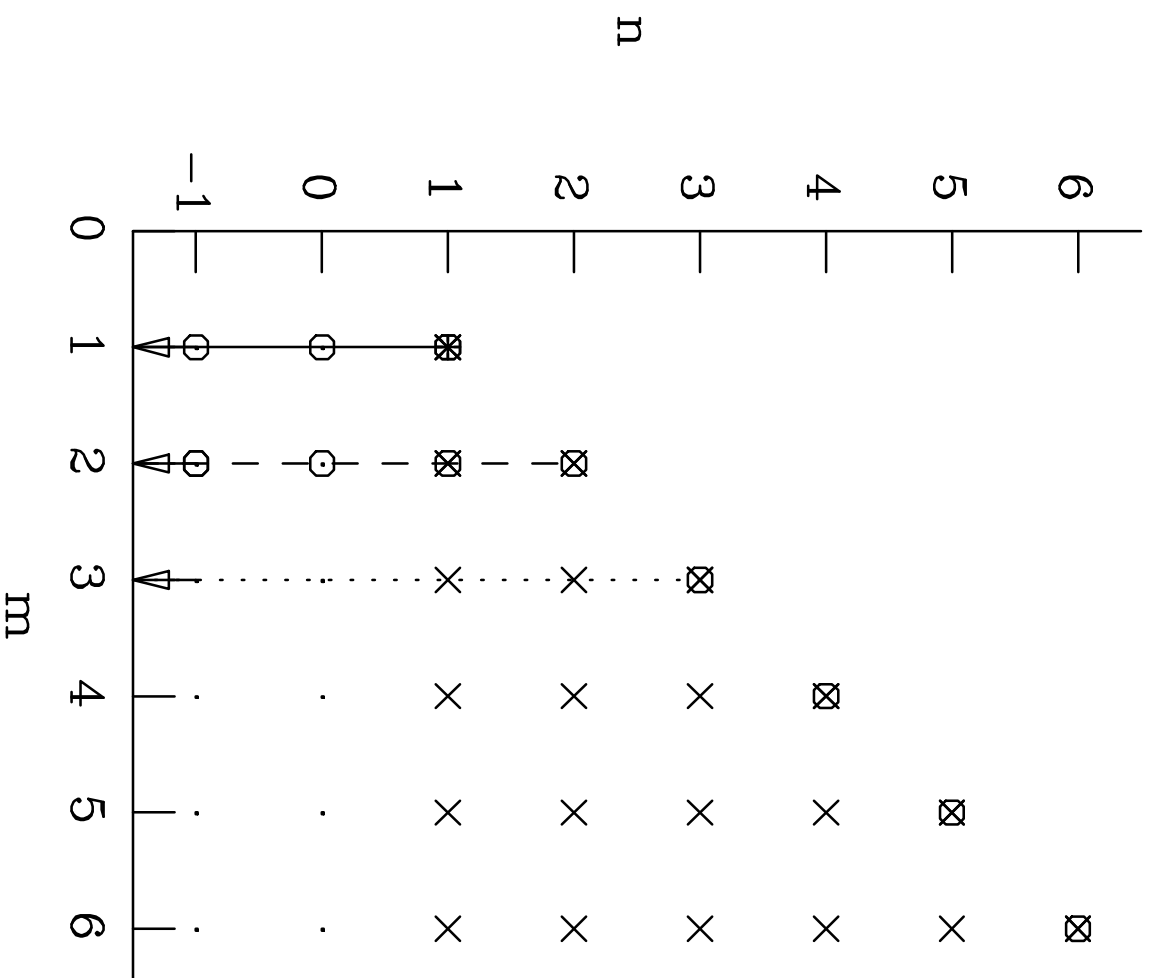


Fig. 1a

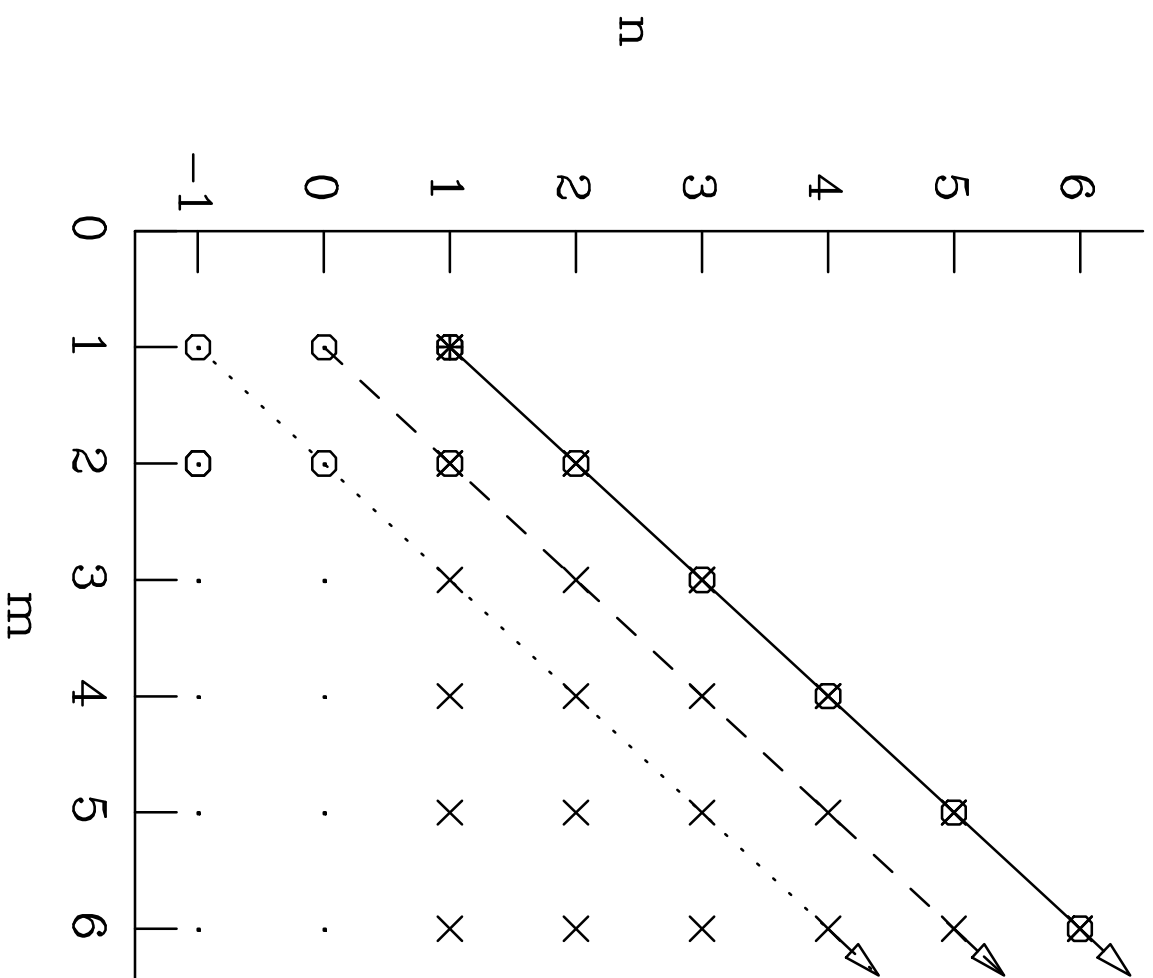


Fig. 1b



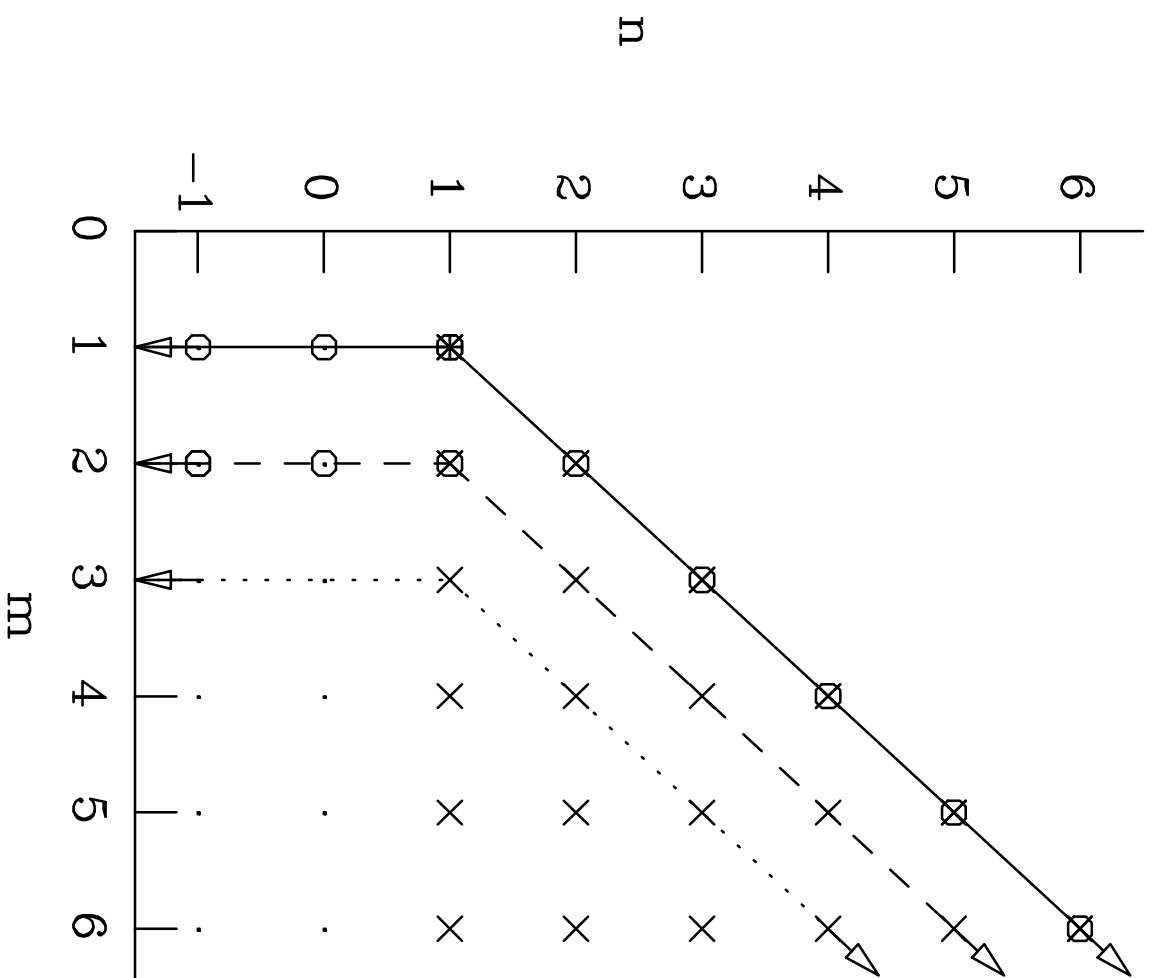


Fig. 1c

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$\xi$

Fig. 2a

R

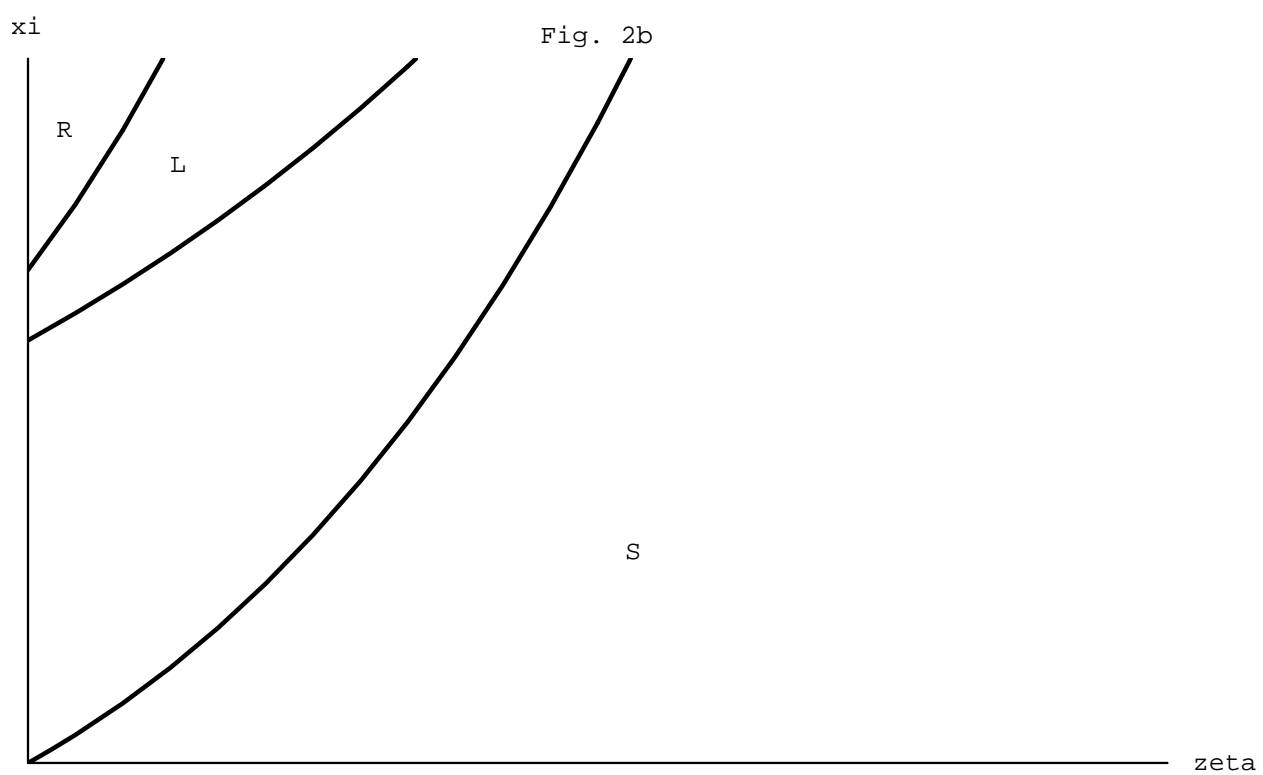
L

S

$\zeta$



The diagram shows a coordinate system with a vertical axis labeled  $\xi$  and a horizontal axis labeled  $\zeta$ . A vertical line segment on the  $\xi$ -axis is labeled R. A diagonal line starting from the origin and extending upwards and to the right is labeled L. The region between the  $\zeta$ -axis and the diagonal line L is labeled S. The region above the diagonal line L is labeled R.



$\xi$

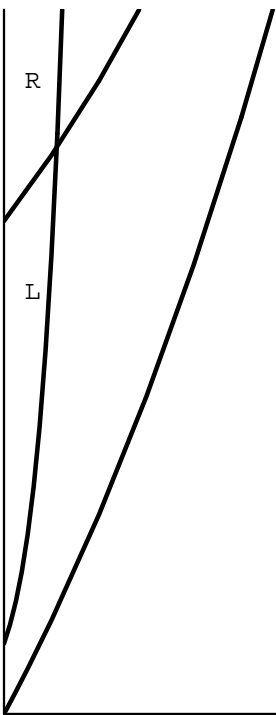


Fig. 2c

$S$

$\zeta$

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Fig. 3a  
 $\ln G$

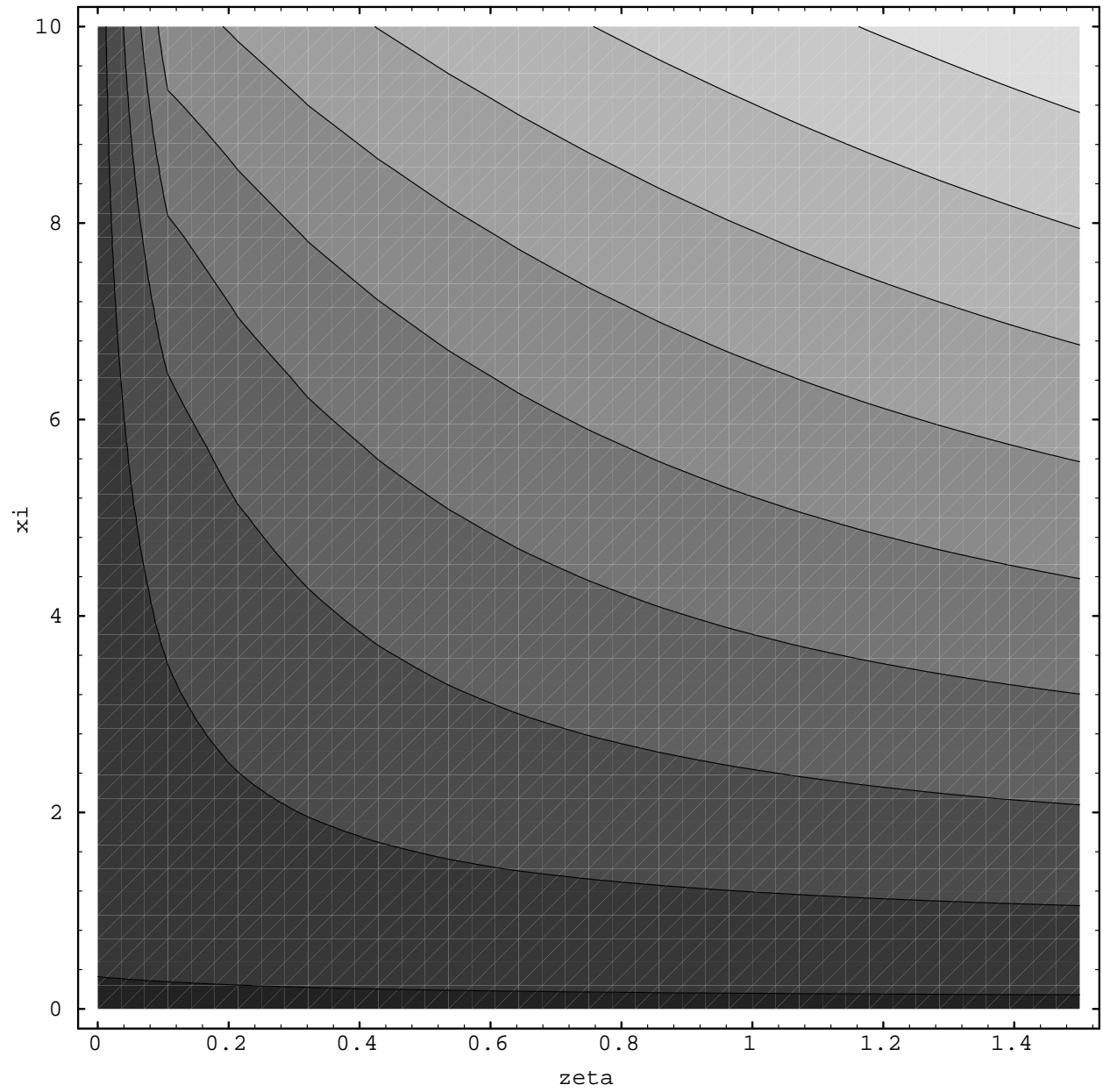


Fig. 3b  
 $\ln F_2$

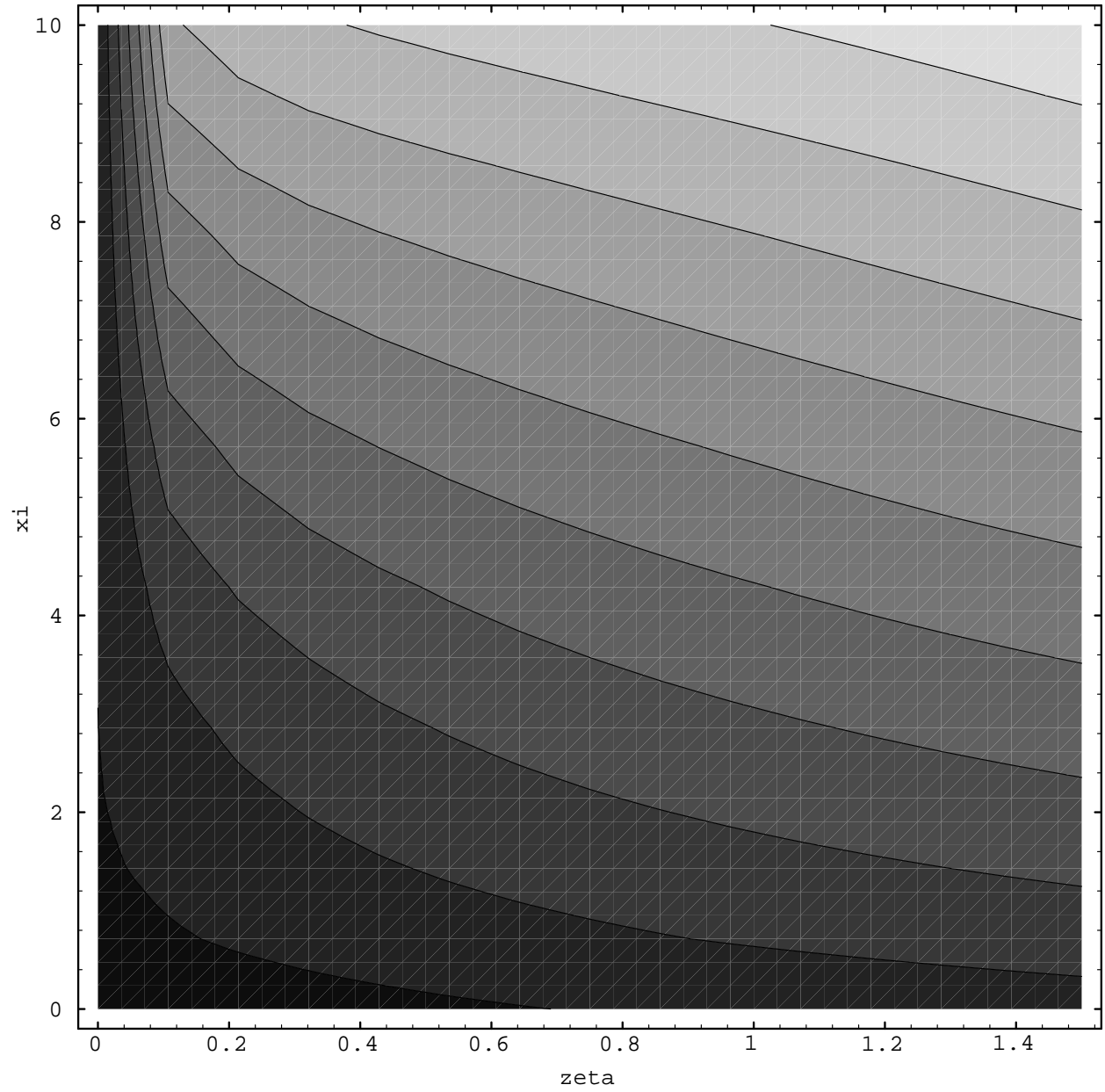
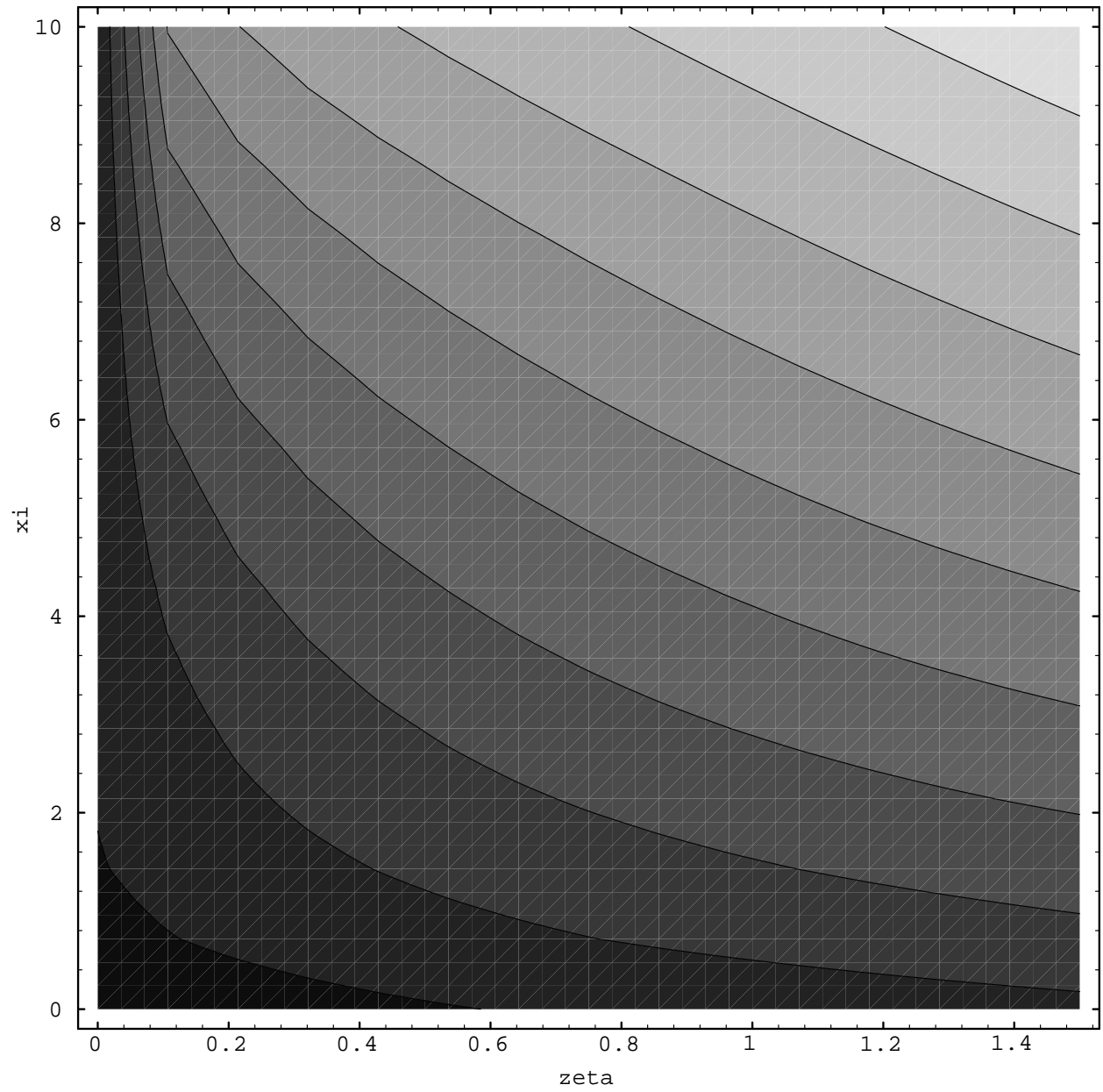




Fig. 3c  
 $\ln F_2$



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